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Quantisation in indefinite metric

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Abstract. The general properties of the indefinite metric Fock quantisation are studied. Some applications of the abstract construction and examples (including the four-dimensional pure gauge model) are discussed.

1. Introduction

The recent interest in gauge quantum field theories has raised some fundamental questions about the structural properties of those theories. In particular it has been proved (Strocchi 1977) that the use of an indefinite metric is an unavoidable feature if one wants to preserve locality and relativistic covariance. The general scheme of quantisation in the Fock space (given originally by Fock (1932) and analysed and further developed by Cook (1953) and Segal (1956, 1959, 1961)) requires therefore a generalisation to an indefinite metric.

The main purpose of this paper is to give a general construction of the Fock quantisation with respect to indefinite inner products of a certain type (see § 1) defined on a Hilbert space $\mathscr{H}$. The obtained field operators are in general not essentially self-adjoint as in the positive metric case, but only symmetric with respect to the corresponding inner products. Therefore the Weyl form of the commutation relations is realised by operators which are unitary with respect to some inner product, but in general not unitary and not even bounded.

Applying the above construction, one can define in a precise way the Fock representations of free fields (in general, not only of Wightman type) defined by arbitrary two-point functions (not necessarily positive definite), which are only required to be translation-invariant, Lorentz-covariant and obey the spectral condition. The notion of these representations is necessary if one wants to use the corresponding fields in order to build Lagrangian models and to treat them by perturbation theory.

Some of the above fields have an interesting feature—they violate the cluster decomposition property (for the definition see, for example, Bogolubov *et al* (1975, p 272)) and as a consequence produce confining potentials. To be concrete let us consider the following example. It is well-known that in the three-dimensional Euclidean space E^3, the Coulomb and Yukawa potentials are fundamental solutions of the operators Δ and $\Delta - m^2$ $(m > 0)$ respectively, i.e.

$$\Delta \frac{1}{r} = -4\pi\delta(\boldsymbol{x}) \tag{1.1}$$

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$$(\Delta - m^2) \frac{e^{-mr}}{r} = -4\pi\delta(\boldsymbol{x}) \tag{1.2}$$

where $\boldsymbol{x} = (x^1, x^2, x^3) \in E^3$, $r = [\Sigma_{i=1}^3 (x^i)^2]^{1/2} = |\boldsymbol{x}|$ and $\Delta = \Sigma_{i=1}^3 \partial^2/\partial x^{i2}$. The corresponding free fields in the four-dimensional Minkowski space M^4 (with metric tensor $g_{\mu\nu}$, (μ, $\nu = 0, \ldots, 3$) and signature $(+---)$) are the electromagnetic potential A_μ and the scalar field φ with mass m, which obey the equations

$$\Box A_\mu = 0 \qquad (\Box + m^2)\varphi = 0 \tag{1.3}$$

where $x = (x^0, \boldsymbol{x}) \in M^4$ and $\Box = g_{\mu\nu}(\partial/\partial x_\mu)(\partial/\partial x_\nu)$ (the summation convention is used). Likewise

$$\Delta^2 r = -8\pi\delta(\boldsymbol{x}) \tag{1.4}$$

and analogously to the above the corresponding relativistic field satisfies

$$\Box^2 \phi = 0. \tag{1.5}$$

In this sense the field ϕ produces a linear potential and therefore a constant force, i.e. the expected type of interaction between quarks. Besides the trivial constant solution and the massless scalar field, there are two other relativistic fields which obey the last equation. These are the dipole ghost field and the field with two-point Wightman function proportional to $x^2 = g_{\mu\nu}x^\mu x^\nu$, which we will call in the following the harmonic field. As we mentioned, the general scheme which will be worked out allows an explicit and rigorous treatment of such types of fields. For models containing as building blocks the dipole and harmonic fields and having a non-trivial S matrix we refer the reader to d'Emilio and Mintchev (1979).

The paper is organised in the following way. In the first section we recall some definitions and results from the theory of indefinite metric spaces (see Bognár (1974) and references therein). In the second section we formulate the quantisation procedure. Section 3 contains the main theorems concerning the properties of the obtained quantised fields. As an illustration of the abstract construction, in § 4 some examples are discussed. The last section contains applications of the above examples. We consider the general solution of the four-dimensional pure gauge model. Two particular solutions of this model are studied by Kleiber (1965) and Zwanziger (1978). Unfortunately they do not exhibit a characteristic feature of the general solution, namely the failure of the cluster decomposition property, which is deeply connected with a possible confining mechanism. For this reason, in our opinion the general solution mimics an expected property of non-Abelian gauge theories.

We adopt the following notations. By $\mathbb{R}^1$ and $\mathbb{C}^1$ we denote the real line and the complex plane respectively, $\mathbb{R}^n$ and $\mathbb{C}^n$ are the corresponding Cartesian products and $\mathscr{S}(\mathbb{R}^n)$ is the space of complex Schwartz test functions on $\mathbb{R}^n$. The symbols $\otimes$ and $\oplus$ stand for the tensor product and the direct sum. By $\mathscr{D}_A$ and $\mathscr{R}_A$ we denote respectively the domain and the range of the operator A. We shall deal with linear operators defined on linear subspaces (in general not closed) of a Hilbert space $\mathscr{H}$.

2. Some definitions and preliminary results

Let $\mathscr{H}$ be a separable complex Hilbert space with topology τ generated by the scalar product $(\cdot, \cdot)$. An inner product on $\mathscr{H}$ is a complex-valued function $\langle \cdot, \cdot \rangle$ defined for all pairs $\varphi, \psi \in \mathcal{H}$, such that the conditions

$$\langle \varphi, \alpha\psi + \beta\chi \rangle = \alpha\langle \varphi, \psi \rangle + \beta\langle \varphi, \chi \rangle \tag{2.1a}$$

$$\langle \varphi, \psi \rangle = \overline{\langle \psi, \varphi \rangle} \tag{2.1b}$$

are fulfilled for each $\alpha, \beta \in \mathbb{C}^1$ and $\varphi, \psi, \chi \in \mathcal{H}$, where the bar means complex conjugation. The inner product $\langle \cdot, \cdot \rangle$ is called non-degenerate iff $\langle \varphi, \psi \rangle = 0$ for all $\psi \in \mathcal{H}$ implies $\varphi = 0$. We denote by $\mathcal{I}(\mathcal{H})$ the set of all non-degenerate inner products which are jointly τ- continuous, i.e. $\forall \varphi, \psi \in \mathcal{H}$

$$|\langle \varphi, \psi \rangle| \leq c\|\varphi\| \, \|\psi\| \tag{2.2}$$

where c is some real positive constant and $\|\cdot\|$ is the norm corresponding to the scalar product $(\cdot, \cdot)$. Let $i(\mathcal{H}) \subset \mathcal{I}(\mathcal{H})$ be the subset characterised by the condition $c \leq 1$ in (2.2). It is easy to see that $i(\mathcal{H})$ is a convex subset of $\mathcal{I}(\mathcal{H})$. The following proposition and corollary characterise completely the sets $\mathcal{I}(\mathcal{H})$ and $i(\mathcal{H})$.

Proposition 1. The inner product $\langle \cdot, \cdot \rangle$ belongs to $\mathcal{I}(\mathcal{H})$ iff there exists a linear invertible bounded self-adjoint operator η, such that for all pairs $\varphi, \psi \in \mathcal{H}$

$$\langle \varphi, \psi \rangle = (\eta\varphi, \psi). \tag{2.3}$$

The operator η corresponding to $\langle \cdot, \cdot \rangle \in \mathcal{I}(\mathcal{H})$ is unique†.

The proof follows immediately from the theorem on the representation of jointly continuous non-degenerate sesquilinear form over a Hilbert space (see Akhiezer and Glasman 1966, p42). We stress that in general η^{-1} is unbounded, but has dense domain $\mathcal{D}_{\eta^{-1}} = \mathcal{R}_\eta$ and is self-adjoint (Rudin 1973, p334).

Corollary. $\langle \cdot, \cdot \rangle \in i(\mathcal{H})$ iff the corresponding metric operator η is a contraction on $\mathcal{H}$, i.e. $\|\eta\| \leq 1$.

Let us fix $\langle \cdot, \cdot \rangle \in \mathcal{I}(\mathcal{H})$. The operator A with dense domain $\mathcal{D}_A (\overline{\mathcal{D}}_A = \mathcal{H})$ is called $\langle \cdot, \cdot \rangle$ −symmetric iff $\forall \varphi, \psi \in \mathcal{D}_A$

$$\langle A\varphi, \psi \rangle = \langle \varphi, A\psi \rangle. \tag{2.4}$$

Proposition 2. Every $\langle \cdot, \cdot \rangle$ −symmetric operator is closable.

Proof. Let A be a $\langle \cdot, \cdot \rangle$ −symmetric operator and let the sequence $\{\varphi_n\} \subset \mathcal{D}_A$ have the properties

$$\operatorname*{s-lim}_{n\to\infty} \varphi_n = 0 \qquad \operatorname*{s-lim}_{n\to\infty} A\varphi_n = \psi$$

(by s-lim we denote the strong limit in $\mathcal{H}$). In order to prove the statement of the proposition we have to show that $\psi = 0$. Indeed, for every $\chi \in \mathcal{D}_A$ one has

$$\langle \chi, \psi \rangle = \lim_{n\to\infty} \langle \chi, A\varphi_n \rangle = \lim_{n\to\infty} \langle A\chi, \varphi_n \rangle = 0.$$

Now, using the fact that $\overline{\mathcal{D}}_A = \mathcal{H}$ and that the inner product is non-degenerate, we conclude that $\psi = 0$.

† In the mathematical literature η is called the Gram operator of $\langle \cdot, \cdot \rangle$ with respect to $(\cdot, \cdot)$. Here we shall follow the physical terminology and shall call η the metric operator.

Let A be a densely defined operator on $\mathscr{D}_A(\overline{\mathscr{D}}_A = \mathscr{H})$ and let

$$\mathscr{D}_{A^{\circledast}} = \{\psi \in \mathscr{H} | \exists \chi \in \mathscr{H}: \langle \psi, A\varphi \rangle = \langle \chi, \varphi \rangle \forall \varphi \in \mathscr{D}_A\}. \tag{2.5}$$

The vector χ from (1.5) is uniquely determined by ψ, because $\langle \cdot , \cdot \rangle$ is non-degenerate and $\overline{\mathscr{D}}_A = \mathscr{H}$. Therefore one can define the operator $A^{\circledast}$ with domain $\mathscr{D}_{A^{\circledast}}$ by

$$A^{\circledast}\psi = \chi. \tag{2.6}$$

The operator $A^{\circledast}$ is called the $\langle \cdot , \cdot \rangle$-adjoint off A. In analogy with the positive metric case, the densely defined operator A is called $\langle \cdot , \cdot \rangle$-self-adjoint if $A = A^{\circledast}$.

The operator U with dense domain $\mathscr{D}_U$ and range $\mathscr{R}_U(\overline{\mathscr{D}}_U = \overline{\mathscr{R}}_U = \mathscr{H})$ is called $\langle \cdot , \cdot \rangle$-unitary iff $\forall \varphi, \psi \in \mathscr{D}_U$

$$\langle U\varphi, U\psi \rangle = \langle \varphi, \psi \rangle. \tag{2.7}$$

Proposition 3. Every $\langle \cdot , \cdot \rangle$-unitary operator is a closable linear operator. It has an inverse, which is also $\langle \cdot , \cdot \rangle$-unitary. For the proof see Bracci *et al* (1975).

3. Quantisation

In this section a general construction for quantisation in indefinite metric is given. We start with a complex separable Hilbert space $\{\mathscr{H}^{(1)}, (\cdot , \cdot)_{(1)}\}$, called the one-particle space. As usual the Fock space $\mathscr{F}(\mathscr{H}^{(1)})$ over $\mathscr{H}^{(1)}$ is defined by

$$\mathscr{F}(\mathscr{H}^{(1)}) = \bigoplus_{n=0}^{\infty} \mathscr{H}^{(n)} \tag{3.1}$$

where $\mathscr{H}^{(0)} = \mathbb{C}^1$ and $\mathscr{H}^{(n)}$ (for $n > 0$) is the tensor product $\otimes_{k=1}^{n} \mathscr{H}_k^{(1)}$. By $V^{(n)} \subset \mathscr{H}^{(n)}$ $(n > 0)$ we denote the total subset† of decomposable vectors

$$V^{(n)} = \{\varphi_1 \otimes \ldots \otimes \varphi_n | \varphi_i \in \mathscr{H}^{(1)}, i = 1, \ldots , n\}. \tag{3.2}$$

Let $\langle \cdot , \cdot \rangle_{(1)}$ be an arbitrary but fixed element of $i(\mathscr{H}^{(1)})$ with metric operator η. We define $\Gamma(\eta)$ to be the operator on $\mathscr{F}(\mathscr{H}^{(1)})$ which equals $\otimes_{k=1}^{n} \eta$ when restricted to $\mathscr{H}^{(n)}$ for $n > 0$, and which equals the identity on $\mathscr{H}^{(0)}$.

Proposition 4. The inner product defined by

$$\langle \varphi, \psi \rangle = (\Gamma(\eta)\varphi, \psi), \qquad \varphi, \psi \in \mathscr{F}(\mathscr{H}^{(1)}), \tag{3.3}$$

where $(\cdot , \cdot)$ is the scalar product in $\mathscr{F}(\mathscr{H}^{(1)})$, belongs to $i(\mathscr{F}(\mathscr{H}^{(1)}))$.

Proof. Because of the corollary of proposition 1, η is a self-adjoint contraction on $\mathscr{H}^{(1)}$, which implies that $\Gamma(\eta)$ is well defined and is a self-adjoint contraction on $\mathscr{F}(\mathscr{H}^{(1)})$.

In order to prove that the inner product (3.3) is non-degenerate, it is sufficient to verify that this is true on $\mathscr{H}^{(n)}$ for arbitrary but fixed n. On

$$\mathscr{R}_\eta^{(n)} = \{\varphi_1 \otimes \ldots \otimes \varphi_n | \varphi_i \in \mathscr{R}_\eta, i = 1, \ldots , n\}$$

we define the operator $\otimes_{k=1}^{n} \eta^{-1}$ by

$$\bigotimes_{k=1}^{n} \eta^{-1} : \varphi_1 \otimes \ldots \otimes \varphi_n \mapsto \eta^{-1}\varphi_1 \otimes \ldots \otimes \eta^{-1}\varphi_n$$

† $V \subset \mathscr{H}$ is called total iff the set of finite linear combinations of elements of V, denoted by $L(V)$, is dense in $\mathscr{H}$.

and extend it by linearity on $L(\mathscr{R}_\eta^{(n)})$, which is dense in $\mathscr{H}^{(n)}$, because $\mathscr{R}_\eta$ is dense in $\mathscr{H}^{(1)}$ (see proposition 1)†. It is easily seen that $\otimes_{k=1}^n \eta^{-1}$ is the inverse of $\otimes_{k=1}^n \eta$ on $L(\mathscr{R}_\eta^{(n)})$ and is symmetric and therefore closable.

Let $\langle\varphi, \chi\rangle = 0 \, \forall \chi \in \mathscr{H}^{(n)}$, where $\varphi \in \mathscr{H}^{(n)}$. This implies $\otimes_{k=1}^n \eta\varphi = 0$. There exists a sequence $\{\varphi_m\} \subset L(\mathscr{R}_\eta^{(n)})$, such that

$$\operatorname*{s-lim}_{m\to\infty} \varphi_m = \varphi.$$

Consider the sequence $\{\psi_m = \otimes_{k=1}^n \eta\varphi_m\} \subset L(\mathscr{R}_\eta^{(n)})$. Using that $\otimes_{k=1}^n \eta$ is bounded we obtain

$$\operatorname*{s-lim}_{m\to\infty} \psi_m = \operatorname*{s-lim}_{m\to\infty} \bigotimes_{k=1}^n \eta\varphi_m = \bigotimes_{k=1}^n \eta\Big(\operatorname*{s-lim}_{m\to\infty} \varphi_m\Big) = \bigotimes_{k=1}^n \eta\varphi = 0 \tag{3.4a}$$

and, on the other hand,

$$\operatorname*{s-lim}_{m\to\infty} \bigotimes_{k=1}^n \eta^{-1}\psi_m = \operatorname*{s-lim}_{m\to\infty} \varphi_m = \varphi. \tag{3.4b}$$

From equations (3.4) and the fact that $\otimes_{k=1}^n \eta^{-1}$ is closable we obtain $\varphi = 0$, which completes the proof of the statement.

Let $\varphi \in \mathscr{H}^{(1)}$ be fixed. We define the map

$$b^-(\varphi): V^{(n)} \to V^{(n-1)} \qquad (n > 0)$$

by

$$b^-(\varphi)\varphi_1 \otimes \ldots \otimes \varphi_n = \langle\varphi, \varphi_1\rangle_{(1)}\varphi_2 \otimes \ldots \otimes \varphi_n \tag{3.5a}$$

and $b^-(\varphi)\mathscr{H}^{(0)} = 0$. For $n \geq 0$ we define also the map

$$b^+(\varphi): V^{(n)} \to V^{(n+1)}$$

by

$$b^+(\varphi)\varphi_1 \otimes \ldots \otimes \varphi_n = \varphi \otimes \varphi_1 \otimes \ldots \otimes \varphi_n. \tag{3.5b}$$

Proposition 5. The operators $b^\pm(\varphi)$ satisfy the conditions:

(a) $$\|b^\pm(\varphi)\psi\| \leq \|\varphi\|_{(1)}\|\psi\| \qquad \forall \psi \in V^{(n)} \tag{3.6}$$

where $\|\cdot\|_{(1)}$ and $\|\cdot\|$ are the norms in $\mathscr{H}^{(1)}$ and $\mathscr{F}(\mathscr{H}^{(1)})$ respectively;

(b) $b^+(\varphi)$ is the $\langle\cdot,\cdot\rangle$-adjoint of $b^-(\varphi)$.

Proof. (a) Using that $\langle\cdot,\cdot\rangle_{(1)} \in i(\mathscr{H}^{(1)})$ we obtain for all $\psi = \psi_1 \otimes \ldots \otimes \psi_n \in V^{(n)}$

$$\|b^-(\varphi)\psi\|^2 = |\langle\varphi, \psi_1\rangle_{(1)}|^2\|\psi_2 \otimes \ldots \otimes \psi_n\|^2 \leq \|\varphi\|_{(1)}^2\|\psi_1\|_{(1)}^2\|\psi_2 \otimes \ldots \otimes \psi_n\|^2 = \|\varphi\|_{(1)}^2\|\psi\|^2.$$

The inequality for $b^+(\varphi)$ can be proved analogously.

(b) This condition follows from the fact that $\forall \psi \in V^{(n)}, \forall \chi \in V^{(n+1)}$ one has

$$\begin{aligned}\langle b^+(\varphi)\psi, \chi\rangle &= \langle b^+(\varphi)\psi_1 \otimes \ldots \otimes \psi_n, \chi_1 \otimes \ldots \otimes \chi_{n+1}\rangle \\ &= \langle\varphi, \chi_1\rangle_{(1)}\langle\psi_1, \chi_2\rangle_{(1)} \ldots \langle\psi_n, \chi_{n+1}\rangle_{(1)}\end{aligned}$$

† The operator $\otimes_{k=1}^n \eta^{-1}$ cannot be extended in general on $\mathscr{H}^{(n)}$, because η^{-1} is in general unbounded.

and on the other hand

$$\langle \psi, b^-(\varphi)\chi\rangle = \langle \psi_1 \otimes \ldots \otimes \psi_n, b^-(\varphi)\chi_1 \otimes \ldots \otimes \chi_{n+1}\rangle$$
$$= \langle \varphi, \chi_1\rangle_{(1)}\langle \psi_1, \chi_2\rangle_{(1)} \ldots \langle \psi_n, \chi_{n+1}\rangle_{(1)}.$$

Corollary. $b^\pm(\varphi)$ can be extended on $\mathscr{H}^{(n)}$ as bounded operators of norm $\|\varphi\|_{(1)}$.

We shall describe in detail only the boson (symmetric) quantisation. The case of fermion (antisymmetric) quantisation can be considered analogously. The n-particle space $\mathscr{H}_s^{(n)}$ of the boson Fock space can be defined by

$$\mathscr{H}_s^{(n)} = S_n \mathscr{H}^{(n)}$$

where S_n are the symmetrisation operators:

$$S_0 = 1 \qquad S_n = \frac{1}{n!}\sum_{\sigma\in\mathscr{P}_n} \sigma \qquad (n>0) \tag{3.7}$$

($\mathscr{P}_n$ is the permutation group of n elements). Then the symmetric Fock space $\mathscr{F}_s(\mathscr{H}^{(1)})$ is given by

$$\mathscr{F}_s(\mathscr{H}^{(1)}) = \bigoplus_{n=0}^{\infty} \mathscr{H}_s^{(n)}. \tag{3.8}$$

On $\mathscr{H}_s^{(n)}$ and $\mathscr{H}_s^{(n+1)}$ respectively, we introduce the creation and annihilation operators $a^+(\varphi)$ and $a^-(\varphi)$ as follows:

$$a^+(\varphi) = (n+1)^{1/2} S_{n+1} b^+(\varphi) \tag{3.9a}$$

$$a^-(\varphi) = (n+1)^{1/2} b^-(\varphi). \tag{3.9b}$$

The result of proposition 5(a) and the fact that S_n are orthogonal projections imply the estimates

$$\|a^+(\varphi)\psi\| \leq (n+1)^{1/2}\|\varphi\|_{(1)}\|\psi\| \qquad \forall\psi\in\mathscr{H}_s^{(n)} \tag{3.10a}$$

$$\|a^-(\varphi)\psi\| \leq (n+1)^{1/2}\|\varphi\|_{(1)}\|\psi\| \qquad \forall\psi\in\mathscr{H}_s^{(n+1)} \tag{3.10b}$$

which are the same as in the positive metric case (Reed and Simon 1975).

It is easy to see that the symmetrisation operators S_n, defined by (3.7), are not only self-adjoint but also $\langle\cdot,\cdot\rangle$-self-adjoint. Combining this with proposition 5(b) we obtain that $a^+(\varphi)$ is the $\langle\cdot,\cdot\rangle$-adjoint of $a^-(\varphi)$.

Let $F_0 \subset \mathscr{F}_s(\mathscr{H}^{(1)})$ be the finite particle subspace, i.e. $\varphi = (\varphi^{(0)}, \ldots, \varphi^{(n)}, \ldots)$ (where $\varphi^{(n)} \in \mathscr{H}_s^{(n)}$) belongs to F_0 iff there exists a number n_φ such that $\forall n > n_\varphi$, $\varphi^{(n)} = 0$. Clearly $\bar{F}_0 = \mathscr{F}_s(\mathscr{H}^{(1)})$.

The operators $a^\pm(\varphi)$ can be extended by linearity to F_0. Then the quantised field $\Phi(\varphi)$ is defined on F_0 by

$$\Phi(\varphi) = \frac{1}{\sqrt{2}}[a^+(\varphi) + a^-(\varphi)]. \tag{3.11}$$

The real linear map†

$$\varphi \mapsto \Phi(\varphi) \tag{3.12}$$

† We remark that (3.12) is not a complex linear map, because $\varphi \mapsto a^-(\varphi)$ is antilinear.

will be called quantisation over $\mathscr{H}^{(1)}$ with respect to $\langle\cdot,\cdot\rangle_{(1)}$. The operator $\Phi(\varphi)$ is $\langle\cdot,\cdot\rangle$-symmetric on F_0, but in general not symmetric. Nevertheless, as we shall show in the next section the field Φ possesses a dense set of analytic vectors and preserves the main properties known for the standard positive metric Fock quantisation.

The conclusion from the above considerations is that for each (complex separable) Hilbert space $\mathscr{H}^{(1)}$, there is in general an infinite family of quantised fields, which are in one-to-one correspondence with the elements of $i(\mathscr{H}^{(1)})$. Obviously the well-known Segal field $\Phi_s(\varphi)$, which is obtained by quantisation over $\mathscr{H}^{(1)}$ with respect to $(\cdot,\cdot)_{(1)}$, is an element of this family because $(\cdot,\cdot)_{(1)}\in i(\mathscr{H}^{(1)})$.

4. The main theorems

Let us fix $\langle\cdot,\cdot\rangle_{(1)}\in i(\mathscr{H}^{(1)})$. The following theorems give the fundamental properties of the quantisation $\varphi\mapsto\Phi(\varphi)$ over $\mathscr{H}^{(1)}$ with respect to $\langle\cdot,\cdot\rangle_{(1)}$.

Theorem 1. (a) The operator $\Phi(\varphi)$ is closable $\forall\varphi\in\mathscr{H}^{(1)}$;

(b) F_0 is a set of analytic vectors for $\Phi(\varphi)$;

(c) If $\{\varphi_k\}\subset\mathscr{H}^{(1)}$ and $\operatorname*{s-lim}_{k\to\infty}\varphi_k=\varphi$, then

$$\operatorname*{s-lim}_{k\to\infty}\Phi(\varphi_k)\psi=\Phi(\varphi)\psi\qquad\forall\psi\in F_0;\tag{4.1}$$

(d) The vacuum vector $\Omega=\{1,0,\ldots,0,\ldots\}\in F_0$ is cyclic, i.e. the set $\{\Phi(\varphi_1)\ldots\Phi(\varphi_n)\Omega|\varphi_i\in\mathscr{H}^{(1)}, i=1,\ldots,n; n=1,2,\ldots\}$ is total in $\mathscr{F}_s(\mathscr{H}^{(1)})$;

(e) For each $\chi\in F_0$ and $\varphi,\psi\in\mathscr{H}^{(1)}$

$$\Phi(\varphi)\Phi(\psi)\chi-\Phi(\psi)\Phi(\varphi)\chi=\mathrm{i}\,\mathrm{Im}\,\langle\varphi,\psi\rangle_{(1)}\chi.\tag{4.2}$$

Remark. In the positive metric case from (b) and the fact that $\Phi(\varphi)$ is symmetric, by Nelson's analytic vector theorem it follows that $\Phi(\varphi)$ is essentially self-adjoint.

Proof. (a) The statement follows from proposition 2, because as mentioned before, $\Phi(\varphi)$ is $\langle\cdot,\cdot\rangle$-symmetric on F_0, which is dense in $\mathscr{F}_s(\mathscr{H}^{(1)})$, and $\langle\cdot,\cdot\rangle$ is non-degenerate (see proposition 4).

(b) We have to show that $\forall\psi\in F_0$,

$$\sum_{k=0}^{\infty}\frac{\|\Phi(\varphi)^k\psi\|}{k!}t^k<\infty\tag{4.3}$$

for some $t>0$, which can be done in the same way as in the positive metric case (Reed and Simon 1975) using the estimates (3.10).

(c) Clearly it is sufficient to prove equation (4.1) for $\psi\in\mathscr{H}_s^{(n)}$. From equations (3.10) using that $\varphi\mapsto\Phi(\varphi)$ is a real linear map we obtain

$$\|\Phi(\varphi_k)\psi-\Phi(\varphi)\psi\|=\|\Phi(\varphi_k-\varphi)\psi\|\leqslant[2(n+1)]^{1/2}\|\varphi_k-\varphi\|_{(1)}\|\psi\|$$

so $\operatorname{s-lim}_{k\to\infty}\Phi(\varphi_k)\psi=\Phi(\varphi)\psi$.

(d) The proof goes as in the positive metric case and uses essentially the fact that $\langle\cdot,\cdot\rangle$ is non-degenerate.

(e) Let $\chi \in V^{(n)}$. Using equations (3.5) and (3.9) one easily obtains

$$\Phi(\varphi)\Phi(\psi)\chi - \Phi(\psi)\Phi(\varphi)\chi = \mathrm{i}\,\mathrm{Im}\,\langle\varphi, \psi\rangle_{(1)}\chi. \tag{4.4}$$

The extension of (4.4) by continuity on $\mathscr{H}_s^{(n)}$ and linearity on F_0 gives (4.2).

Corollary. The operator $W(\varphi) = \mathrm{e}^{\mathrm{i}\Phi(\varphi)}$, $\varphi \in \mathscr{H}^{(1)}$, is well defined on F_0 as the strong limit of the corresponding series, i.e. $\forall\psi \in F_0$

$$W(\varphi)\psi = \underset{K\to\infty}{\text{s-lim}} \sum_{k=0}^{K} \frac{[\mathrm{i}\Phi(\varphi)]^k}{k!}\psi. \tag{4.5}$$

In contrast with the positive metric case (where $W(\varphi)$ is a unitary operator) the above-defined operator is in general not even bounded and one has to be careful with questions concerning its domain of definition.

Theorem 2. (a) $W(\varphi)$ is closable $\forall\varphi \in \mathscr{H}^{(1)}$;

(b) For each $\varphi, \psi \in \mathscr{H}^{(1)}, \chi \in F_0$

$$\bar{W}(\varphi+\psi)\chi = \exp\left(\frac{\mathrm{i}}{2}\,\mathrm{Im}\langle\varphi, \psi\rangle_{(1)}\right)\bar{W}(\varphi)\bar{W}(\psi)\chi \tag{4.6}$$

where $\bar{W}$ is the closure of W;

(c) For each $\varphi \in \mathscr{H}^{(1)}$, $W(\varphi)$ is a $\langle\cdot,\cdot\rangle$-unitary operator on F_0.

Remark. Equation (4.6) is known as the Weyl form of the commutation relations. As in the positive metric case, equation (4.2) by itself does not imply (4.6).

Proof. (a) Let $\{\psi_n\} \subset F_0$, $\text{s-lim}_{n\to\infty}\psi_n = 0$ and $\text{s-lim}_{n\to\infty} W(\varphi)\psi_n = \psi$. For any $\chi \in F_0$ one has:

$$\begin{aligned}\langle\chi, \psi\rangle &= \lim_{n\to\infty}\langle\chi, \mathrm{e}^{\mathrm{i}\Phi(\varphi)}\psi_n\rangle = \lim_{n\to\infty}\left[\lim_{K\to\infty}\left\langle\chi, \sum_{k=0}^{K}\frac{[\mathrm{i}\Phi(\varphi)]^k}{k!}\psi_n\right\rangle\right] \\ &= \lim_{n\to\infty}\left[\lim_{K\to\infty}\left\langle\sum_{k=0}^{K}\frac{[-\mathrm{i}\Phi(\varphi)]^k}{k!}\chi, \psi_n\right\rangle\right] = \lim_{n\to\infty}\langle W(-\varphi)\chi, \psi_n\rangle = 0.\end{aligned}$$

Using that $\bar{F}_0 = \mathscr{F}_s(\mathscr{H}^{(1)})$ and that the inner product $\langle\cdot,\cdot\rangle$ is non-degenerate we obtain $\psi = 0$, which proves the statement.

(b) First of all we shall show that $\forall\varphi, \psi \in \mathscr{H}^{(1)}$, the series

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{[\mathrm{i}\Phi(\varphi)]^k[\mathrm{i}\Phi(\psi)]^l}{k!\,l!} \tag{4.7}$$

converges absolutely on F_0. Indeed, $\forall\chi \in \mathscr{H}_s^{(n)}$, equations (3.10) imply the estimate

$$\|\Phi(\varphi)^k\Phi(\psi)^l\chi\| \leqslant 2^{\frac{1}{2}(k+l)}(n+1)^{1/2}\ldots(n+k+l)^{1/2}\|\varphi\|_{(1)}^k\|\psi\|_{(1)}^l\|\chi\|.$$

Therefore

$$\begin{aligned}&\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\|\Phi(\varphi)^k\Phi(\psi)^l\chi\|}{k!\,l!}t^k t^l \\ &\qquad\leqslant \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\sqrt{2}t)^{k+l}\frac{(n+1)^{1/2}\ldots(n+k+l)^{1/2}}{k!\,l!}\|\varphi\|_{(1)}^k\|\psi\|_{(1)}^l\|\chi\|\end{aligned}$$

$$= \|\chi\| \sum_{h=0}^{\infty} (\sqrt{2}t)^h \frac{(n+1)^{1/2} \dots (n+h)^{1/2}}{h!} \sum_{k=0}^{h} \frac{h}{k!(h-k)!} \|\varphi\|_{(1)}^k \|\psi\|_{(1)}^{h-k}$$

$$= \|\chi\| \sum_{h=0}^{\infty} [\sqrt{2}t(\|\varphi\|_{(1)} + \|\psi\|_{(1)})]^h \frac{(n+1)^{1/2} \dots (n+h)^{1/2}}{h!}$$

$$\leqslant \|\chi\| \sum_{h=0}^{\infty} [\sqrt{2}t(\|\varphi\|_{(1)} + \|\psi\|_{(1)})]^h \frac{[(n+h)!]^{1/2}}{h!} < \infty \tag{4.8}$$

because

$$\sum_{h=0}^{\infty} s^h \frac{[(n+h)!]^{1/2}}{h!} < \infty \qquad \forall s \in \mathbb{R}^1.$$

Now the absolute convergence of (4.7) on F_0 follows from the convergence of (4.8), because the vectors of F_0 are finite linear combinations of vectors from $\mathscr{H}_s^{(n)}$ with arbitrary n.

As a second step one proves that $\forall \chi \in F_0$, $W(\psi)\chi \in \mathscr{D}_{\bar{W}(\varphi)}$. Because of equation (4.5)

$$\operatorname*{s-lim}_{L\to\infty} \chi_L = W(\psi)\chi \tag{4.9}$$

where

$$\chi_L = \sum_{l=0}^{L} \frac{[\mathrm{i}\Phi(\psi)]^l}{l!} \chi \in F_0$$

and therefore

$$W(\varphi)\chi_L = \operatorname*{s-lim}_{K\to\infty} \sum_{k=0}^{K} \frac{[\mathrm{i}\Phi(\varphi)]^k}{k!} \chi_L.$$

The limit

$$\operatorname*{s-lim}_{L\to\infty} W(\varphi)\chi_L = \operatorname*{s-lim}_{L\to\infty} \left(\operatorname*{s-lim}_{K\to\infty} \sum_{k=0}^{K} \frac{[\mathrm{i}\Phi(\varphi)]^k}{k!} \chi_L \right)$$

$$= \operatorname*{s-lim}_{L\to\infty} \left(\operatorname*{s-lim}_{K\to\infty} \sum_{k=0}^{K} \sum_{l=0}^{L} \frac{[\mathrm{i}\Phi(\varphi)]^k [\mathrm{i}\Phi(\psi)]^l}{l!k!} \chi \right), \tag{4.10}$$

which exists as a consequence of the absolute convergence of (4.7), is independent of the order in which one takes the two limits and is equal to (4.7). Equation (4.9), combined with the existence of the limit (4.10) and the fact that W is closable, imply $\forall \chi \in F_0$, $\forall t \in \mathbb{R}^1$

$$\bar{W}(t\varphi)\bar{W}(t\psi)\chi = \bar{W}(t\varphi)W(t\psi)\chi = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{[\mathrm{i}t\Phi(\varphi)]^k [\mathrm{i}t\Phi(\psi)]^l}{k!l!}. \tag{4.11}$$

On the other hand $\forall \chi \in F_0$, $\forall t \in \mathbb{R}^1$

$$\exp\left(-\frac{\mathrm{i}}{2} \operatorname{Im}\langle t\varphi, t\psi\rangle_{(1)}\right) \bar{W}(t\varphi + t\psi)\chi$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k!l!} \left[-\frac{\mathrm{i}t^2}{2} \operatorname{Im}\langle \varphi, \psi\rangle_{(1)} \right]^k [\mathrm{i}t\Phi(\varphi + \psi)]^l \chi \tag{4.12}$$

because χ is an analytic vector for $\Phi(\varphi+\psi)$. Finally, using equation (4.2), equation (4.6) follows from the term-by-term comparison of the convergent power series (4.11) and (4.12).

(c) Using that Φ is $\langle\cdot,\cdot\rangle$-symmetric on F_0 and the result of point (b) above, we obtain $\forall\psi, \chi\in F_0, \forall\varphi\in\mathscr{H}^{(1)}$

$$\begin{aligned}\langle W(\varphi)\psi, W(\varphi)\chi\rangle &= \lim_{K\to\infty}\lim_{L\to\infty}\left\langle \sum_{k=0}^{K}\frac{[\mathrm{i}\Phi(\varphi)]^k}{k!}\psi, \sum_{l=0}^{L}\frac{[\mathrm{i}\Phi(\varphi)]^l}{l!}\chi\right\rangle \\ &= \lim_{K\to\infty}\lim_{L\to\infty}\left\langle\psi, \sum_{k=0}^{K}\sum_{l=0}^{L}\frac{[-\mathrm{i}\Phi(\varphi)]^k[\mathrm{i}\Phi(\varphi)]^l}{k!l!}\chi\right\rangle \\ &= \langle\psi, \bar{W}(-\varphi)\bar{W}(\varphi)\chi\rangle = \langle\psi,\chi\rangle.\end{aligned}$$

Finally we shall discuss the realisation of symmetries in the above scheme. Let U be a $\langle\cdot,\cdot\rangle_{(1)}$-unitary operator in $\mathscr{H}^{(1)}$ and let

$$\mathscr{D}_U^{(n)} = \{\varphi_1\otimes\ldots\otimes\varphi_n|\varphi_i\in\mathscr{D}_U, i=1,\ldots,n\} \tag{4.13a}$$

$$\mathscr{R}_U^{(n)} = \{\varphi_1\otimes\ldots\otimes\varphi_n|\varphi_i\in\mathscr{R}_U, i=1,\ldots,n\}. \tag{4.13b}$$

The operator $\Gamma(U)$, defined on $L(S_n\mathscr{D}_U^{(n)})$ ($L(S_n\mathscr{D}_U^{(n)})$ means the set of finite linear combinations of elements of $S_n\mathscr{D}_U^{(n)}$ and S_n is given by equation (3.7)) by $\otimes_{k=1}^n U$ ($n>0$) and on $\mathscr{H}^{(0)}$ as the identity, can be continued by linearity to the whole

$$F_0^d = \{\varphi=(\varphi^{(0)},\ldots,\varphi^{(n)},\ldots)\in F_0|\varphi^{(k)}\in L(S_k\mathscr{D}_U^{(k)})\}$$

as a $\langle\cdot,\cdot\rangle$-unitary operator. Analogously we can define on

$$F_0^r = \{\varphi=(\varphi^{(0)},\ldots,\varphi^{(n)},\ldots)\in F_0|\varphi^{(k)}\in L(S_k\mathscr{R}_U^{(k)})\}$$

the operator $\Gamma(U^{-1})$†.

Proposition 6. Let U be a $\langle\cdot,\cdot\rangle_{(1)}$-unitary operator in $\mathscr{H}^{(1)}$. Then $\forall\varphi\in\mathscr{D}_U, \forall\psi\in F_0^r$

$$\Gamma(U)\Phi(\varphi)\Gamma(U^{-1})\psi = \Phi(U\varphi)\psi \tag{4.14}$$

where Φ is the field obtained by quantisation over $\mathscr{H}^{(1)}$ with respect to $\langle\cdot,\cdot\rangle_{(1)}$.

Proof. It is sufficient to prove equation (4.14) for $\psi\in S_n\mathscr{R}_U^{(n)}$. For the annihilation operator one has

$$\begin{aligned}&\Gamma(U)a^-(\varphi)\Gamma(U^{-1})\psi \\ &\quad= \Gamma(U)a^-(\varphi)\Gamma(U^{-1})S_n\psi_1\otimes\ldots\otimes\psi_n \\ &\quad= \frac{1}{n!}\Gamma(U)a^-(\varphi)\Gamma(U^{-1})\sum_{\sigma\in\mathscr{P}_n}\psi_{\sigma(1)}\otimes\ldots\otimes\psi_{\sigma(n)} \\ &\quad= \frac{1}{n!}\Gamma(U)a^-(\varphi)\sum_{\sigma\in\mathscr{P}_n}U^{-1}\psi_{\sigma(1)}\otimes\ldots\otimes U^{-1}\psi_{\sigma(n)} \\ &\quad= \frac{\sqrt{n}}{n!}\Gamma(U)\sum_{\sigma\in\mathscr{P}_n}\langle\varphi, U^{-1}\psi_{\sigma(1)}\rangle_{(1)}U^{-1}\psi_{\sigma(2)}\otimes\ldots\otimes U^{-1}\psi_{\sigma(n)}\end{aligned}$$

† In the case when U and U^{-1} are bounded, $\Gamma(U)$ and $\Gamma(U^{-1})$ are well defined by continuity on $\mathscr{H}_s^{(n)}$ and therefore on F_0.

$$=\frac{\sqrt{n}}{n!}\sum_{\sigma\in\mathcal{P}_n}\langle\varphi, U^{-1}\psi_{\sigma(1)}\rangle_{(1)}\psi_{\sigma(2)}\otimes\ldots\otimes\psi_{\sigma(n)}$$

$$=\frac{\sqrt{n}}{n!}\sum_{\sigma\in\mathcal{P}_n}\langle U\varphi, \psi_{\sigma(1)}\rangle_{(1)}\psi_{\sigma(2)}\otimes\ldots\otimes\psi_{\sigma(n)}$$

$$=a^{-}(U\varphi)\psi.$$

For the creation operator the proof goes in the same way.

5. Some examples

In order to apply the general scheme from the previous sections for the reconstruction of free fields from their two-point functions, one has to do one preliminary step. The point is that knowing the two-point vacuum expectation value $w(x_1, x_2)$ which is a generalised function over a certain set of test functions $\mathcal{T}$ ($\mathcal{T}=\mathcal{S}(\mathbb{R}^n)$ for Wightman fields), we have to find a triplet† $\{F, \mathcal{H}^{(1)}, \eta\}$, where F is a continuous map

$$F:\mathcal{T}\to\mathcal{H}^{(1)} \tag{5.1a}$$

such that $\forall f, g\in\mathcal{T}$

$$\int\int \bar{f}(x_1)w(x_1, x_2)g(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2=(F(f), \eta F(g))_{(1)} \tag{5.1b}$$

and η is the metric operator. Then we can use the general construction for quantisation over $\mathcal{H}^{(1)}$ with respect to $\langle\cdot,\cdot\rangle_{(1)}=(\eta\,\cdot,\cdot)_{(1)}$.

Now let us explain the above reconstruction procedure on the example of the harmonic fields $\phi_{h\pm}$ with two-point Wightman functions

$$w_{\pm}(x_1, x_2)=\pm(x_1-x_2)^2 \qquad x_1, x_2\in M^4‡.$$

We start with the space $\mathbb{C}^6$ with scalar product§

$$(z, u)_{(1)}=2\sum_{\mu=0}^{3}\bar{z}_\mu u_\mu+\bar{z}_4u_4+\bar{z}_5u_5 \tag{5.2}$$

where $z=(z_0,\ldots,z_3, z_4, z_5)$, $u=(u_0,\ldots,u_3, u_4, u_5)\in\mathbb{C}^6$. In terms of (5.2), two non-degenerate indefinite inner products are defined by

$$\langle z, u\rangle_{(1)\pm}=(z, \eta_\pm u)_{(1)} \tag{5.3}$$

where $\eta_\pm$ are 6×6 matrices with the following block matrix form

$$\eta_\pm=\mp\begin{pmatrix} g & & 0 \\ & 0 & 1 \\ 0 & 1 & 0\end{pmatrix}. \tag{5.4}$$

Obviously $\eta_\pm^2=\mathbb{1}_6$ ($\mathbb{1}_6$ being the 6×6 unit matrix) and therefore

$$|\langle z, u\rangle_{(1)\pm}|\leqslant\|z\|_{(1)}\|u\|_{(1)}.$$

† From the generalisation of the reconstruction theorem to indefinite metric (Yngvason 1977) it is known that the Hilbert structure $\{F, \mathcal{H}^{(1)}, \eta\}$ is not uniquely determined by $w(x_1, x_2)$.

‡ Because of the fact that the metric is indefinite, two signs for the two-point function are possible.

§ Obviously this scalar product generates the standard topology on $\mathbb{C}^6$.

Let $\Phi_{\pm}$ be the fields obtained by quantisation over $\mathbb{C}^6$ with respect to (5.3). Then the Wightman fields $\phi_{h\pm}$, for which we are looking, are defined on $\mathcal{S}(\mathbb{R}^4)$ by

$$\phi_{h\pm}(f)=\Phi_{\pm}/\sqrt{2}\,z_{\hat{f}} \qquad f\in\mathcal{S}(\mathbb{R}^4) \tag{5.5}$$

where

$$z_{\hat{f}}=(\partial_\mu\hat{f}(0),\Box\hat{f}(0),\hat{f}(0))\in\mathbb{C}^6, \tag{5.6}$$

$\hat{f}(p)$ is the Fourier transform of $f(x)$ given by

$$\hat{f}(p)=\int \mathrm{e}^{\mathrm{i}px}f(x)\,\mathrm{d}^4x \qquad px=p_\mu x_\nu g^{\mu\nu} \tag{5.7}$$

and $\partial_\mu=\partial/\partial p^\mu$, $\Box=g_{\mu\nu}(\partial/\partial p^\mu)(\partial/\partial p^\nu)$. Indeed let us compute the two-point Wightman functions. We have

$$\begin{aligned}
\langle\phi_{h\pm}(f)\Omega,\,\phi_{h\pm}(g)\Omega\rangle_{\pm}
&=\langle\Phi_{\pm}(\sqrt{2}z_{\hat{f}})\Omega,\,\Phi_{\pm}(\sqrt{2}z_{\hat{g}})\Omega\rangle_{\pm}\\
&=\mp2\partial_\mu\bar{\hat{f}}(0)\,\partial^\mu\hat{g}(0)\mp\Box\bar{\hat{f}}(0)\hat{g}(0)\mp\bar{\hat{f}}(0)\Box\hat{g}(0)\\
&=\mp\int\Box[\bar{\hat{f}}(p)\hat{g}(p)]\delta(p)\,\mathrm{d}^4p=\pm\int\int\bar{f}(x_1)(x_1-x_2)^2g(x_2)\,\mathrm{d}^4x_1\,\mathrm{d}^4x_2
\end{aligned} \tag{5.8}$$

which shows that the map F (see equation (5.1a)) is given by

$$F(f)=z_{\hat{f}} \qquad f\in\mathcal{S}(\mathbb{R}^4).$$

All the other Wightman functions can be written in a standard way in terms of the two-point functions (5.8).

The constructed fields obey the commutation relations

$$[\phi_{h\pm}(x_1),\,\phi_{h\pm}(x_2)]=0 \qquad \forall x_1,x_2\in M^4\dagger. \tag{5.9}$$

Let us consider the representation $U(a,\Lambda)$ of the Poincaré group $\mathcal{P}=\mathbb{R}^4\,.\,\mathrm{O}(3,1)$. Its form is suggested by equation (5.6) and the transformation law

$$\hat{f}(p)\mapsto\mathrm{e}^{\mathrm{i}pa}\hat{f}(\Lambda^{-1}p).$$

In general, the representation $U(a,\Lambda)$ is defined on $\mathcal{H}^{(1)}=\mathbb{C}^6$ by

$$U(a,\Lambda)z=(z_\nu{\Lambda^{-1}}^\nu{}_\mu+\mathrm{i}a_\mu z_5,\,z_4-a^2z_5+2\mathrm{i}z_\nu{\Lambda^{-1}}^\nu{}_\mu a^\mu,\,z_5) \tag{5.10}$$

and is $\langle\cdot,\cdot\rangle_{(1)\pm}$-unitary, i.e. $\forall z,u\in\mathcal{H}^{(1)}$, $\forall(a,\Lambda)\in\mathcal{P}$

$$\langle U(a,\Lambda)z,\,U(a,\Lambda)u\rangle_{(1)\pm}=\langle z,u\rangle_{(1)\pm}. \tag{5.11}$$

The operators $U(a,\Lambda)$ are not unitary, but are bounded ($\mathcal{H}^{(1)}$ is finite dimensional) and therefore the operators $\Gamma(U(a,\Lambda))$ and $\Gamma(U(a,\Lambda)^{-1})$ are well defined on F_0, and

$$\Gamma(U(a,\Lambda))\phi_{h\pm}(f(x))\Gamma(U(a,\Lambda)^{-1})=\phi_{h\pm}(f(\Lambda^{-1}(x-a))). \tag{5.12}$$

It is important to stress that $\mathcal{H}^{(1)}$ contains the following subspace $\mathcal{V}^{(1)}=\{z\in\mathcal{H}^{(1)}|z_\mu=z_5=0\}$ of vectors fixed with respect to $\{U(a,1)\}_{a\in\mathbb{R}^1}$. It gives rise to an infinite dimensional subspace $\mathcal{V}\subset\mathcal{F}_s(\mathcal{H}^{(1)})$ of vectors fixed with respect to

† Because of this commutation property, all time-ordered Green functions, including one type of field (ϕ_{h+} or ϕ_{h-}), coincide with the corresponding Wightman functions.

$\{\Gamma(U(a, 1))\}_{a\in\mathbb{R}^1}$. Therefore, besides the mathematical vacuum Ω, there is an infinite dimensional space of translation invariant states, called physical vacuums.

We remark also that the n-particle space $\mathcal{H}^{(n)}$ of the above described field is finite dimensional (more precisely $6n$-dimensional) which simplifies the construction.

Considering the quantisation of the dipole field, we choose a slightly different formulation from the existent explicit Fock realisations (Ferrari 1974, Zwanziger 1978). Some of the details of our derivation are important for the better understanding of this quantisation problem. In order to define the one-particle space, let us consider the direct sum

$$L^2\left(V_+, \frac{d^3p}{|\boldsymbol{p}|}\right) \oplus L^2\left(V_+, \frac{d^3p}{|\boldsymbol{p}|}\right) \qquad |\boldsymbol{p}| = \left(\sum_{i=1}^{3} p_i^2\right)^{1/2} \tag{5.13}$$

where V_+ is the future light cone $V_+ = \{p \in M^4 | p^2 = 0, p_0 \geq 0\}$ and $d^3p/|\boldsymbol{p}|$ is the Lorentz-invariant measure on it. The elements of the above space will be denoted by

$$L^2 \oplus L^2 \ni \varphi(p) = \begin{pmatrix} \varphi_1(p) \\ \varphi_2(p) \end{pmatrix}.$$

Besides the standard scalar product

$$(\varphi, \psi)_{L^2\oplus L^2} = \int_{V_+} \frac{d^3p}{|\boldsymbol{p}|} \varphi^*(p)\psi(p) \tag{5.14}$$

where the asterisk means Hermitian conjugation, we introduce also the scalar product

$$(\varphi, \psi) = \int_{V_+} \frac{d^3p}{|\boldsymbol{p}|} \varphi^*(p) N\psi(p) \tag{5.15}$$

where N is the 2×2 matrix

$$N = \frac{1}{\sqrt{5}} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}. \tag{5.16}$$

It is easily seen that

$$\frac{1}{5^{1/4}} \|\varphi\|_{L^2\oplus L^2} \leq \|\varphi\| \leq \frac{2}{5^{1/4}} \|\varphi\|_{L^2\oplus L^2}$$

so that $(\cdot, \cdot)$ generates the same complete topology as $(\cdot, \cdot)_{L^2\oplus L^2}$. In terms of (5.15), two indefinite inner products are defined by

$$\langle\varphi, \psi\rangle_\pm = (\varphi, \eta_\pm\psi) \tag{5.17}$$

with metric operators $\eta_\pm$ given by

$$\eta_\pm = \pm\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ -2 & -1 \end{pmatrix}. \tag{5.18}$$

Obviously $\eta_\pm^2 = \mathbb{1}$, where $\mathbb{1}$ is the unit operator in $L^2 \oplus L^2$.

Let $\Phi_\pm$ be the fields obtained by quantisation over $L^2 \oplus L^2$ with respect to the inner products (5.17). Then the dipole ghost fields $\phi_{d\pm}$ are defined on $\mathcal{S}_0(\mathbb{R}^4) = \{f(x) \in \mathcal{S}(\mathbb{R}^4) | \int f(x)\, d^4x = 0\}$ by

$$\phi_{d\pm}(f) = \Phi_\pm\left(\frac{1}{(2\pi)^{3/2}} \varphi_f\right) \tag{5.19}$$

where

$$\varphi_f(p) = \begin{pmatrix} \dfrac{1}{np}\hat{f}(p) \\ \\ n\partial\hat{f}(p) \end{pmatrix}. \tag{5.20}$$

In the last formula n is an arbitrary but fixed time-like vector ($n^2>0$), $n\partial = n^\mu\partial/\partial p^\mu$ and $\hat{f}$ is the Fourier transform (5.7). It can be verified that $\varphi_f(p)$ is indeed an element of $L^2 \oplus L^2$. By direct computation for the two-point functions one has

$$\begin{aligned}
&\langle \phi_{d\pm}(f)\Omega, \phi_{d\pm}(g)\Omega\rangle_\pm \\
&= \langle \Phi_\pm\Big(\frac{1}{(2\pi)^{3/2}}\varphi_f\Big)\Omega, \Phi_\pm\Big(\frac{1}{(2\pi)^{3/2}}\varphi_g\Big)\Omega\rangle_\pm \\
&= \pm\frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{2|\boldsymbol{p}|}\frac{1}{np}\Big(\frac{1}{np} - n\partial\Big)\bar{\hat{f}}(p)\hat{g}(p) \\
&= \pm\int 2\pi\theta(p_0)\delta^1(p^2)\bar{\hat{f}}(p)\hat{g}(p)\frac{\mathrm{d}^4p}{(2\pi)^4} \\
&= \pm\int\int \bar{f}(x_1)w(x_1, x_2)g(x_2)\,\mathrm{d}^4x_1\,\mathrm{d}^4x_2\dagger
\end{aligned} \tag{5.21}$$

where

$$w(x_1, x_2) = -\frac{1}{(4\pi)^2}\ln[-(x_1-x_2)^2 + \mathrm{i}\epsilon(x_1^0 - x_2^0)] \tag{5.22a}$$

which has to be regarded as a functional on $\mathscr{S}_0(\mathbb{R}^4)$. Its continuation on $\mathscr{S}(\mathbb{R}^4)$ is given by

$$\frac{1}{(4\pi)^2}\ln\frac{l^2}{-(x_1-x_2)^2+\mathrm{i}\epsilon(x_1^0-x_2^0)} \tag{5.22b}$$

where l is an arbitrary positive length.

The map F (see equation (5.1a)) now is given by

$$F(f) = \frac{1}{2^2\pi^{3/2}}\varphi_f \qquad f \in \mathscr{S}_0(\mathbb{R}^4).$$

The dipole field is local, since

$$\begin{aligned}
[\phi_{d\pm}(x_1), \phi_{d\pm}(x_2)] &= \pm[w(x_1, x_2) - w(x_2, x_1)] \\
&= \pm\frac{\mathrm{i}}{8\pi}\epsilon(x_1^0 - x_2^0)\theta((x_1-x_2)^2) = \pm\mathrm{i}E(x_1-x_2)
\end{aligned} \tag{5.23}$$

where the function $E(x)$ defined by the last equation has the properties

$$\begin{aligned}
&\Box^2 E(x) = \frac{\partial^2}{\partial x^{02}}E(x)\Big|_{x^0=0} = \frac{\partial}{\partial x^0}E(x)\Big|_{x^0=0} = E(x)\Big|_{x^0=0} = 0 \\
&\frac{\partial}{\partial x^0}\Box E(x)\Big|_{x^0=0} = \delta(\boldsymbol{x}).
\end{aligned} \tag{5.24}$$

† For $n=(1, \mathbf{0})$ one obtains the definition of $\theta(p_0)\delta^1(p^2)$, given by Vladimirov (1966, p294).

It can be easily verified that the canonical commutation relations (CCR) following in the standard way from the Lagrangians

$$\mathscr{L}_{\pm} = \mp(\partial_\mu \phi_{\pm d}\partial^\mu \chi + \tfrac{1}{2}\chi\chi) \tag{5.25}$$

are coexistent with (5.23). Indeed, using the Lagrange–Euler equations

$$\begin{aligned} \Box\phi_{\pm d} &= \chi \\ \Box\chi &= 0 \end{aligned} \tag{5.26}$$

as well as the properties of the function $E(x)$ and equation (5.23), we obtain

$$\left[\frac{\partial}{\partial x_1^0}\phi_{\pm d}(x_1), \chi(x_2)\right]\bigg|_{x_1^0=x_2^0} = \pm \mathrm{i}\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)$$

$$\left[\frac{\partial}{\partial x_1^0}\chi(x_1), \phi_{\pm d}(x_2)\right]\bigg|_{x_1^0=x_2^0} = \pm \mathrm{i}\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)$$

which are exactly the CCR, corresponding to the Lagrangians (5.25).

We stress that, although the vacuum Ω is cyclic with respect to $\Phi_\pm$ (see theorem 1(d)), this is not true for the field $\phi_{d\pm}$, because of the special form of the vectors φ_f given by (5.20). In order to obtain cyclicity of the vacuum for $\phi_{d\pm}$, we define the dipole one-particle space by

$$\mathscr{H}_d^{(1)} = \overline{\{\varphi_f(p) | f \in \mathscr{S}_0(\mathbb{R}^4)\}} \tag{5.27}$$

where the bar means the closure in $L^2 \oplus L^2$.

The $\langle\cdot,\cdot\rangle_\pm$-unitary representation $U(a, \Lambda)$ of the Poincaré group $\mathscr{P}$ is given on the dense (in $\mathscr{H}_d^{(1)}$) set of vectors of the type φ_f by

$$U(a, \Lambda)\varphi_f = \varphi_{f'} \tag{5.28}$$

where $\hat{f}'(p) = \mathrm{e}^{\mathrm{i}pa}\hat{f}(\Lambda^{-1}p)$. The above representation is not unitary. For example, for the translations one has

$$\begin{aligned}(U(a, 1)\varphi_{\hat{f}}, U(a, 1)\varphi_{\hat{g}}) &\\ = (\varphi_{\hat{f}}, \varphi_{\hat{g}}) + \frac{2}{\sqrt{5}}\int_{V_+}\frac{\mathrm{d}^3p}{|\boldsymbol{p}|}&[(an)^2\bar{\hat{f}}(p)\hat{g}(p) - \mathrm{i}(an)\bar{\hat{f}}(p)n\,\partial\hat{g}(p) \\ &+\mathrm{i}(an)\hat{g}(p)n\,\partial\bar{\hat{f}}(p)]\end{aligned} \tag{5.29}$$

where $an = a_\mu n_\nu g^{\mu\nu}$.

We note finally that in order to give a probability interpretation for the models in which the fields described above enter, in analogy with quantum electrodynamics, we have to fix in the total Hilbert space of the model a subspace $\mathscr{H}'$, on which $\langle\cdot,\cdot\rangle$ is positive semi-definite and such that

$$\Omega \in \mathscr{H}'$$

$$U(a, \Lambda)\mathscr{H}' \subset \mathscr{H}' \qquad \forall (a, \Lambda) \in \mathscr{P}.$$

For the general axioms of a quantum field theory with indefinite metric we refer the reader to the paper of Strocchi (1977). We shall discuss in the next section the choice of $\mathscr{H}'$ in the particular case of the pure gauge model and shall see that the dipole and harmonic fields realise the interaction, but do not give rise to physical asymptotic states.

The last property seems to be a general feature of the $\Box^2$ type interaction and can be used as confinement mechanism in the quark-gluon system (see d'Emilio and Mintchev 1979).

6. The four-dimensional pure gauge model

Let us consider the Lagrangian of Zwanziger (1978).

$$\mathscr{L} = -B\,\partial_\mu A^\mu + \tfrac{1}{2}B^2 + I^\mu(A_\mu - \partial_\mu\phi) - \tilde{\psi}\gamma^\mu\left(\tfrac{1}{2}\mathrm{i}\,\overleftrightarrow{\partial}_\mu + gA_\mu\right)\psi + m\tilde{\psi}\psi \tag{6.1}$$

where the fields are varied independently and the symbol $\sim$ means Dirac conjugation. The equations of motion are

$$\begin{aligned} &\partial_\mu\phi = A_\mu \qquad \partial_\mu A^\mu = B \\ &I_\mu = g\tilde{\psi}\gamma_\mu\psi - \partial_\mu B \qquad \partial_\mu I^\mu = 0 \\ &\gamma^\mu(\mathrm{i}\,\partial_\mu + gA_\mu)\psi - m\psi = 0. \end{aligned} \tag{6.2}$$

Because of the conservation of the spinor current $g\tilde{\psi}\gamma_\mu\psi$, the above system implies

$$\gamma^\mu(\mathrm{i}\,\partial_\mu + g\,\partial_\mu\phi)\psi - m\psi = 0 \tag{6.3a}$$

$$\Box^2\phi = 0. \tag{6.3b}$$

The solution of (6.3) can be expressed in terms of the free spinor field ψ_0 with mass m, defined on the Hilbert space $\mathscr{H}_{\psi_0}$, and the field ϕ in the following way

$$\psi = \psi_0 {:}\mathrm{e}^{\mathrm{i}g\phi}{:}. \tag{6.4}$$

The normal exponent in the right-hand side of (6.4) is defined by the series

$$:\mathrm{e}^{\mathrm{i}g\phi}:(x) = \sum_{k=0}^{\infty}\frac{(\mathrm{i}g)^k}{k!}:\phi^k:(x) \tag{6.5}$$

where $:\phi^k:$ denotes the kth normal product of the field ϕ (Wightman and Gårding 1964). The vacuum expectation value of arbitrary finite number of such exponents is well defined (Wightman 1967), which allows us to compute the Wightman functions of ψ and $\tilde{\psi}$. The general solution of equation (6.3b) can be written as a real linear combination of the fields:

(a) $\phi = \phi_0$, the free massless scalar field ($\Box\phi_0 = 0$) (Kleiber 1965);

(b) $\phi = \phi_{d\pm}$, the dipole fields (the case ϕ_{d+} is studied by Zwanziger (1978);

(c) $\phi = \phi_{h\pm}$, the harmonic fields, defined in § 4.

Let us consider the last two cases. When $\phi = \phi_{d\pm}$, the $2n$-point Wightman functions of the fields ψ and $\tilde{\psi}$ have the form

$$\begin{aligned} &w_\pm(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}) \\ &\quad = \langle\Omega, \psi(x_1)\ldots\psi(x_n)\tilde{\psi}(x_{n+1})\ldots\tilde{\psi}(x_{2n})\Omega\rangle \\ &\quad = (l^2)^{\pm n(g/4\pi)^2}(\Omega_{\psi_0}, \psi_0(x_1)\ldots\psi_0(x_n)\tilde{\psi}_0(x_{n+1})\ldots\tilde{\psi}_0(x_{2n})\Omega_{\psi_0}) \\ &\quad\quad \times\frac{\prod_{1\leq i<j\leq n}(x_{ij}^2 x_{n+i\,n+j}^2)^{\pm(g/4\pi)^2}}{\prod_{1\leq i\leq n},\prod_{1\leq j\leq n},(\Delta^+_{i\,n+j})^{\pm(g/4\pi)^2}} \end{aligned} \tag{6.6}$$

where $\Omega_{\psi_0} \in \mathscr{H}_{\psi_0}$ is the vacuum state of the free spinor field, l is the parameter introduced in equation (5.22*b*) and

$$x_{ij}=x_i-x_j \tag{6.7a}$$

$$\Delta_{ij}^{+}=-(x_i-x_j)^2+\mathrm{i}\epsilon(x_i^0-x_j^0). \tag{6.7b}$$

There are two types of transformations which imply independently gauge transformations of the first kind for the field ψ. These are the gauge transformation of the first kind for the field ψ_0

$$\psi_0\mapsto\psi_0\,\mathrm{e}^{\mathrm{i}g\alpha}\qquad\tilde{\psi}_0\mapsto\tilde{\psi}_0\,\mathrm{e}^{-\mathrm{i}g\alpha}\qquad\alpha\in\mathbb{R}^1$$

and the scalar gauge transformation of the field ϕ

$$\phi\mapsto\phi+\alpha\qquad\alpha\in\mathbb{R}^1.$$

The functions (6.6) are invariant under all of these transformations. In the massless case ($m=0$) they are also conformal-invariant and the fields ψ, $\tilde{\psi}$ have anomalous dimension

$$d'=\frac{3}{2}+\left(\frac{g}{4\pi}\right)^2.$$

The physical subspaces $\mathcal{H}'_{d\pm}\subset\mathcal{H}_{\psi_0}\otimes\mathcal{F}_s(\mathcal{H}_d^{(1)})$ ($\mathcal{H}_d^{(1)}$ is given by equation (5.27)) are defined by the Gupta–Bleuler-type conditions

$$\Psi\in\mathcal{H}'_{d\pm}\Leftrightarrow\Box\phi_{d\pm}^{-}(\varphi)\Psi=0\qquad\forall\varphi\in\mathcal{H}_d^{(1)}.$$

It can be shown (Zwanziger 1978) that the factor spaces $\mathcal{H}'_{d\pm}/\mathcal{H}''_{d\pm}$, where $\mathcal{H}''_{d\pm}=\{\Psi\in\mathcal{H}'_{d\pm}:\langle\Psi,\Psi\rangle=0\}$, are isomorphic to $\mathcal{H}_{\psi_0}$.

It is important to stress that the Wightman functions w_+ and w_- in the case $m>0$ obey the cluster property $\forall g\in\mathbb{R}^1$, in other words for any spacelike vector $a\in M^4$ and for any λ, $g\in\mathbb{R}^1$,

$$\lim_{\lambda\to\infty}w(x_1,\ldots,x_j,x_{j+1}+\lambda a,\ldots,x_{2n}+\lambda a)-w(x_1,\ldots,x_j)w(x_{j+1},\ldots,x_{2n})=0. \tag{6.8}$$

This is not true for w_- when $m=0$. Indeed, for example

$$w_-(x_1,x_2)=\frac{1}{8\pi^2}\gamma^{\mu}x_{12_\mu}\frac{(x_{12}^2)^{(g/4\pi)^2}}{(\Delta_{12}^+)^2}$$

and for sufficiently large g the cluster property is violated.

In the case (c), for the vacuum expectation values of the fields ψ, $\tilde{\psi}$ one has

$$\begin{aligned}w_\pm(x_1,\ldots,x_n,x_{n+1},\ldots,x_{2n})&\\=(\Omega_{\psi_0},\psi_0(x_1)\ldots\psi_0(x_n)\tilde{\psi}_0(x_{n+1})\ldots\tilde{\psi}_0(x_{2n})\Omega_{\psi_0})&\\\times\exp\left[\mp g^2\left(\sum_{1\le i<j\le n}x_{ij}^2+\sum_{n+1\le i<j\le 2n}x_{ij}^2-\sum_{i=1}^{n}\sum_{j=n+1}^{2n}x_{ij}^2\right)\right].&\end{aligned} \tag{6.9}$$

The above functions obey the axiom of positive definiteness and therefore one can define the physical subspaces by

$$\mathcal{H}'_{h\pm}=\overline{\{\mathcal{P}(\psi,\tilde{\psi})\Omega\}}$$

where $\mathcal{P}$ is an arbitrary polynomial and the bar means the closure in $\mathcal{H}_{\psi_0}\otimes\mathcal{F}_s\,(\mathbb{C}^6)$.

The functions (6.9) violate the cluster property in both cases $m=0$ and $m>0$. Indeed, let us consider for example the two-point function $w_-(x_1, x_2)$ with $x_2 = x_1 + \lambda a$, $a^2 = -1$. Because of translation invariance one has

$$w_-(x_1, \lambda a + x_1) = w_{\psi_0}(\lambda a)\, \mathrm{e}^{g^2\lambda^2}$$

where w_{ψ_0} is the two-point function of the free spinor field with mass m. For large λ the behaviour of $w_{\psi_0}(\lambda a)$ is determined from the behaviour of the Hankel function $K_1(m\lambda)$. Using the asymptotic formula 8. 451(6) from Gradshteyn and Ryzhik (1965) we obtain

$$w_-(x_1, x_1 + \lambda a) \xrightarrow[\lambda\to\infty]{} \infty$$

which is in contradiction with equation (6.8). It is interesting to remark that if we are separating simultaneously equal numbers of ψ and $\tilde{\psi}$ in equation (6.9), i.e. if we are separating combinations with zero total charge, the limit $\lambda \to \infty$ exists. The reason is that in this case the exponent in the right-hand side of equation (6.9) is independent of λa. The confinement mechanism becomes more transparent in the momentum space. Indeed, let us consider the two-point Green functions

$$\begin{aligned}\tau_\pm(x_1, x_2) &= \langle \Omega_{\psi_0}, T\psi_0(x_1)\tilde{\psi}(x_2)\Omega_{\psi_0}\rangle \exp[\pm g^2(x_1 - x_2)^2] \\ &= \frac{1}{\mathrm{i}} \sum_{n=0}^{\infty} \frac{(\mp g^2)^n}{n!} S^c_{(n)}\,(x_1 - x_2)\end{aligned}$$

where

$$S^c_{(n)}(\xi) = \mathrm{i}\langle \Omega_{\psi_0}, T\psi_0(\xi)\tilde{\psi}_0(0)\Omega_{\psi_0}\rangle(-\xi^2)^n = S^c_{(0)}(\xi)(-\xi^2)^n. \qquad (6.10)$$

Taking the Fourier transform of (6.10) we obtain

$$\begin{aligned}\hat{S}^c_{(n)}(p) &= \int \mathrm{e}^{\mathrm{i}p\xi} S^c_{(n)}(\xi)\, \mathrm{d}^4\xi \\ &= \iiint \mathrm{e}^{\mathrm{i}p\xi}\, \mathrm{e}^{-\mathrm{i}q\xi}\, \mathrm{e}^{-\mathrm{i}l\xi} \frac{m + \not{q}}{(m^2 - q^2 - \mathrm{i}\epsilon)} (\Box_l)^n \delta(l)\, \mathrm{d}^4\xi\, \mathrm{d}^4 l \frac{\mathrm{d}^4 q}{(2\pi)^4} = \Box_p^n \frac{m + \not{p}}{m^2 - p^2 - \mathrm{i}\epsilon}.\end{aligned}$$

From the last equation it follows that $\hat{S}^c_{(n)}(p)$ has a pole of order $2n+1$ for $p^2 = m^2$, which implies that $\hat{\tau}_\pm(p)$ has an essential singularity at this point. Therefore, in the sense of Lehman–Symanzik–Zimmerman, there are not asymptotic spinor states with mass m.

We shall not speculate further about this sort of confinement mechanism in the trivial gauge coupling, realised by the harmonic fields $\phi_{h\pm}$, but we stress that the representation of the translations in the above model is not unitary on the light cone, which is a necessary condition for confinement in the approach of Strocchi (1978).

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